

ON STIELTJES MEAN INTEGRALS

H. S. KALTENBORN

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By

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INTRODUCTION

The original definition of Stieltjes integral has been subjected to various modifications and extensions which have given rise to different types of this integral. This paper will be concerned chiefly with the four types considered by H. L. Smith,¹ which may be grouped in pairs as "ordinary" and "mean" integrals according to the way in which the approximating sum is formed, the two integrals in each group being distinguished by the method used in passing to the limit. The relations between the different modifications of the integral will be considered briefly, then some necessary and sufficient conditions for the existence of the mean integrals will be derived. The validity of some additional elementary properties will be established for the mean integrals; in particular, a theorem corresponding to the substitution theorem for the ordinary integrals will be obtained for the mean integrals. By means of this theorem, and other properties of the integrals, an extended integration by parts theorem and a theorem on the interchange of order in an iterated integral will be established for both the ordinary and mean integrals. It will be shown, finally, that every linear functional operation on the class of functions having discontinuities of the first kind is expressible as a mean integral with respect to a function of bounded variation.

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1. Transactions of the American Mathematical Society, v. 27 (1925), pp. 491-515. Hereafter this paper will be cited as Sm; other references will be given similarly by means of the abbreviations introduced in the bibliography appearing on page 32.



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CHAPTER I

THE STIELTJES INTEGRALS

1. Definitions of the integrals. The method of integration introduced by T. J. Stieltjes (S, p. 71) generalizes the Riemann integral in that it replaces the variable of integration by a function of this variable. The definition of the integral is similar to the definition of the Riemann integral, and may be stated as follows. Let $\sigma = (a=x_0 < x_1 < x_2 < \dots < x_n=b)$ represent a finite subdivision of (a,b) of norm N , $N = \max. (x_i - x_{i-1})$, $(i=1,2, \dots, n)$, and form the sum

$$(1.1) \quad S(f\Delta\varphi) = \sum_{i=1}^n f(\xi_i) [\varphi(x_i) - \varphi(x_{i-1})], \quad (x_{i-1} \leq \xi_i \leq x_i).$$

If the limit of $S(f\Delta\varphi)$ exists as $N \rightarrow 0$, it is called the norm-Stieltjes integral of $f(x)$ with respect to $\varphi(x)$ on (a,b) , and is denoted by the symbol $NS \int_a^b f d\varphi$; i.e.,

$$(1.2) \quad NS \int_a^b f d\varphi = \lim_N \sum_{i=1}^n f(\xi_i) \Delta_i \varphi, \quad (x_{i-1} \leq \xi_i \leq x_i),$$

where $\Delta_i \varphi = \varphi(x_i) - \varphi(x_{i-1})$. It follows from the definition of such a limit that $NS \int_a^b f d\varphi$ exists if for every $\epsilon > 0$ there exists an N_ϵ such that

$$(1.3) \quad \left| NS \int_a^b f d\varphi - \sum_{i=1}^n f(\xi_i) \Delta_i \varphi \right| < \epsilon, \quad (N < N_\epsilon).$$

In his definition of the integral, Stieltjes restricted the integrand $f(x)$ to be continuous and the integrator $\varphi(x)$ to be of bounded variation, thus insuring the existence of the limit of (1.1); but as now used, the term "Stieltjes integral" applies to any limit such as (1.2), and in which $f(x)$ and $\varphi(x)$ are unrestricted.

The above definition has been modified and extended in various ways, and different types of Stieltjes integrals have been obtained. Thus S. Pollard (Po, p. 74) defines a "modified Stieltjes integral" by taking the limit of (1.1) as the subdivision is made successively finer; i.e., by introducing additional points of subdivision in such a way that each subdivision contains all the points of the

preceding.¹ This integral will be referred to as the finer type of Stieltjes integral, and will be denoted by the symbol $FS \int_a^b f d\phi$; so that, by definition,

$$(1.4) \quad FS \int_a^b f d\phi = \lim_F S(f\Delta\phi).$$

The integral exists if for every $\epsilon > 0$ there exists a subdivision σ_ϵ of (a, b) such that

$$(1.5) \quad \left| FS \int_a^b f d\phi - \sum f(\xi_i) \Delta_i \phi \right| < \epsilon, \quad (\sigma > \sigma_\epsilon),$$

where the notation $\sigma_1 > \sigma_2$ indicates that the subdivision σ_1 is finer than the subdivision σ_2 ; i.e., that σ_1 contains all the points of σ_2 and some additional points.

The integrals (1.2) and (1.4) have many properties in common, and it will be found convenient, therefore, to omit the prefix N or F from the integral sign when the property in question is valid for both types of integrals; thus the symbol $S \int_a^b f d\phi$ will indicate that the integral has been obtained from (1.1) by taking the limit in either the N or F sense. Together, these integrals will be referred to as the "ordinary Stieltjes integrals", or more briefly as the "S-integrals."

The definitions (1.2) and (1.4) were modified by H. L. Smith (Sm, pp. 491-492) by the substitution of

$$(1.6) \quad M_i f = \frac{1}{2} [f(x_{i-1}) + f(x_i)]$$

for $f(\xi_i)$ in the approximating sum (1.1). The integrals thus obtained will be referred to here as the "mean Stieltjes integrals", and will be denoted by $NSm \int_a^b f d\phi$ or $FSm \int_a^b f d\phi$ according to the way in which the limit is taken; thus, by definition,

$$(1.7) \quad Sm(f\Delta\phi) = \sum_{i=1}^n M_i f \Delta_i \phi,$$

$$(1.8) \quad NSm \int_a^b f d\phi = \lim_N Sm(f\Delta\phi), \quad FSm \int_a^b f d\phi = \lim_F Sm(f\Delta\phi).$$

The same convention will be followed for the mean integrals as for the S-integrals; i.e., the integrals will be referred to as the Sm-integrals, and the symbols $Sm \int_a^b f d\phi$ will be used if the property in question is valid if the integral is obtained from (1.7) by taking the limit in either the N or F sense.

1. This method of taking the limit is one form of the general method developed by E. H. Moore and H. L. Smith (M).

A further modification in the approximating sum was considered by B. Dushnik (Du, p. 17) who restricted the point ξ_i to be an inner point of (x_{i-1}, x_i) . The modified integrals obtained in this way from (1.2) and (1.4) will be called the S^* -integrals; they are evidently defined by

$$(1.9) \quad S^*(f\Delta\varphi) = \sum_{i=1}^n f(\xi_i) \Delta_i\varphi, \quad (x_{i-1} < \xi_i < x_i),$$

$$(1.10) \quad NS^* \int_a^b f \, d\varphi = \lim_N S^*(f\Delta\varphi), \quad FS^* \int_a^b f \, d\varphi = \lim_F S^*(f\Delta\varphi).$$

Still another variant is that given by W. H. Young (Y, p. 113); viz.,

$$(1.11) \quad \begin{aligned} YS(f\Delta\varphi) = \sum_{i=1}^n \bigg\{ & f(\xi_i) \left[\varphi(x_i - 0) - \varphi(x_{i-1} + 0) \right] + \\ & f(x_{i-1}) \left[\varphi(x_{i-1} + 0) - \varphi(x_{i-1}) \right] + \\ & f(x_i) \left[\varphi(x_i) - \varphi(x_i - 0) \right] \bigg\}, \quad (x_{i-1} < \xi_i < x_i), \end{aligned}$$

$$(1.12) \quad NYS \int_a^b f \, d\varphi = \lim_N YS(f\Delta\varphi), \quad FYS \int_a^b f \, d\varphi = \lim_F YS(f\Delta\varphi),$$

where it is assumed that the integrator possesses right and left-hand limits at every point in the interval of integration. The foregoing definition is a little more general than that given by Young, for he considered only the norm type of integral, and restricted the integrator to be of bounded variation.

In addition to the preceding modifications of the Stieltjes integral, there are other generalizations which are based on more fundamental changes in the definition of this process of integration. One such type is the Lebesgue-Stieltjes integral, which is defined in terms of certain characteristic properties corresponding to those of the ordinary Lebesgue integral (see H₃, p. 186 ff; also, L, p. 271 ff.). Two generalizations in the same direction are given by J. Radon (Ra, pp. 1322-1332), who describes a process of integration for functions of several variables in which the integration is with respect to an "absolute additive" function and is carried out over a set of points rather than over an interval. In his first definition, the integrand is assumed to be uniformly continuous in the range of integration; and in the second, measurable with respect to the integrator. A general form of integral has been devised by P. J. Daniell (Da, pp. 279-280) which includes the LS-integral and the Radon generalizations, as well as the ordinary Stieltjes integral itself, as special cases.

However, none of the general types of integrals mentioned in this paragraph will be considered further in this paper, except that the Daniell extension will be referred to again in §2 in connection with the relations between the S_m - and the S -integrals.

2. Relation between these integrals. The values of the integrals defined in §1 agree in many cases; in fact, if the NS -integral exists, then the other types exist and have the same value. It follows from the definitions that the existence of either S -integral implies the existence (with the same value) of the corresponding S_m -integral, and of the corresponding S^* -integral; and the existence of any N -integral implies the existence of the corresponding F -integral. The YS -integrals also exist and agree with the S -integrals when the latter exist, assuming that the integrator has a right and left-hand limit at every point. For, by rearranging the terms, and making use of the integration by parts theorem (see (IV), §3), we find that

$$\begin{aligned} YS \int_a^b f \, d\varphi &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left\{ -\varphi(x_{i-1} + 0) [f(\xi_i) - f(x_{i-1})] - \right. \\ &\quad \left. \varphi(x_{i-1}) [f(x_i) - f(\xi_i)] + f(b)\varphi(b) - f(a)\varphi(a), \right. \\ &= -S \int_a^b \varphi \, df + [f\varphi]_a^b = S \int_a^b f \, d\varphi, \end{aligned}$$

for both the N and F types of integrals.

On the other hand, the modified integrals exist in many cases where the ordinary integrals do not exist. Thus $f(x)$ and $\varphi(x)$ may not have simultaneous discontinuities if $NS \int_a^b f \, d\varphi$ is to exist, nor simultaneous left- or right-hand discontinuities if $FS \int_a^b f \, d\varphi$ is to exist; while the integral $NSm \int_a^b f \, d\varphi$ exists (as will be shown in §5) if $f(x)$ and $\varphi(x)$ are any two regular functions of bounded variation, and $FSm \int_a^b f \, d\varphi$ exists for any bounded function $f(x)$, having only discontinuities of the first kind, with respect to any function $\varphi(x)$ of bounded variation. Moreover, both mean integrals exist if $\varphi(x) = f(x) = \underline{\text{any}}$ bounded function, as may be seen from the definitions (1.8).

The complete relationship between these integrals may be exhibited as shown on the following page. A plus sign indicates that the integral at the top of the column exists and has the same value as the integral at the left of the row when the latter exists; and a minus sign indicates that the former integral either does not exist or differs from the latter in value.

	N-integral				F-integral			
	S	Sm	S*	YS	S	Sm	S*	YS
S	+	+	+	+	+	+	+	+
Sm	-1	+	-3	-3	-1	+	-3	-3
S*	-2	-4	+	-4	-2	-4	+	-4
YS	-5	-4	-4	+	-5	-4	-4	+

The filling in of the first row is accomplished at once from the statements made above; and the other relations are obtained by means of examples. Examples for cases (1) and (2) have been given by Smith and Dushnik, respectively. The remaining cases are treated in the examples which follow, which are numbered to correspond to the numerals in the table.

(3) Let $f(x) = 0$ or 1 according as $0 \leq x \leq 1/2$ or $1/2 < x \leq 1$; then $FS^* \int_0^1 f \, df = 0$ and $NS^* \int_0^1 f \, df$ does not exist (Du, p. 31); $FYS \int_0^1 f \, df = 1$ and $NYS \int_0^1 f \, df$ does not exist; but $FSm \int_0^1 f \, df = NSm \int_0^1 f \, df = 1/2$.

(4) Let $f(0) = 1$, $f(1) = 2$, and $f(x) = 0$, ($0 < x < 1$); then $FSm \int_0^1 f \, df = 3/2$, $FYS \int_0^1 f \, df = 3$, and $FS^* \int_0^1 f \, df = 0$. The N-integrals agree with the F-integrals in each case since 0 and 1 are points of subdivision by definition.

(5) This follows at once from (4), since the equivalence of the S- and YS-integrals would imply the equivalence of the YS- and Sm-integrals, which is contrary to (4).

The properties of Daniell (Da, pp. 279-280) for the general form of integral may also be used to show that these modified Stieltjes integrals are distinct. For the YS-integrals are LS-integrals and therefore satisfy these conditions by definition, and the S-integrals have these properties as Daniell shows. But the Sm-integrals and the S*-integrals do not satisfy the condition that $\int f_n \, d\phi \rightarrow \int f \, d\phi$ if $f_n \rightarrow f$ monotonically.¹ Consequently none of these integrals are Daniell integrals, and therefore they do not agree with the S- and YS-integrals.

1. An example is given in §5 to show this for the Sm-integrals. The same example may also be used to show that the property does not hold for the S*-integrals either.

If $\varphi(x)$ is of bounded variation, and if $FS \int_a^b f d\varphi$ exists, the FYS-integral may be expressed in the form

$$FYS \int_a^b f d\varphi = FS \int_a^b f d\gamma + \sum_{i=1}^{\infty} f(c_i) [\varphi(c_i + 0) - \varphi(c_i - 0)],$$

where $\gamma(x)$ is the continuous part of $\varphi(x)$, and $c_0, c_1, c_2, \dots$, denote the points of discontinuity of $\varphi(x)$. For let $\varphi(a) = \varphi(a - 0)$, $\varphi(b) = \varphi(b + 0)$, and place $c_0 = x_0 = a$; then

$$\begin{aligned} FYS \int_a^b f d\varphi &= \lim_F \left\{ \sum_{i=1}^n f(\xi_i) [\varphi(x_i - 0) - \varphi(x_{i-1} + 0)] + \right. \\ &\quad \left. \sum_{i=0}^n f(x_i) [\varphi(x_i + 0) - \varphi(x_i - 0)] \right\} \\ &= \lim_F \left\{ \sum_{i=1}^n f(\xi_i) [\gamma(x_i) - \gamma(x_{i-1})] + \right. \\ &\quad \left. \sum_{i=0}^n f(x_i) [\varphi(x_i + 0) - \varphi(x_i - 0)] \right\} \\ &= FS \int_a^b f d\gamma + \sum_{i=0}^{\infty} f(c_i) [\varphi(c_i + 0) - \varphi(c_i - 0)]. \end{aligned}$$

3. Fundamental properties and existence conditions for the ordinary and mean integrals. The elementary properties of the NS-integral have been discussed by various writers (for example, see L, Chap. XI; and Pe) for the case where the integrand is continuous and the integrator is of bounded variation. Additional properties for this case have been established by H. E. Bray (B), while S. Pollard (Po) has treated both the NS- and FS-integrals for any functions for which the integrals are defined. The properties of the S*-integrals have been investigated by B. Dushnik (Du). Corresponding properties of the Sm-integrals will be derived in this paper. Since the Sm-integrals possess many of the properties of the S-integrals, these four types will be discussed simultaneously. The present section will be devoted to their elementary properties and some fundamental conditions for their existence; further properties will be established in Chapter III. The discussion will be concerned primarily with the mean integrals; the ordinary integrals are included chiefly by way of contrast.

The S- and the Sm-integrals possess the following elementary properties, as may be shown directly from the definitions.

(I) If $\int_a^b f \, d\varphi$ exists, then $\int_a^b k f \, d\varphi$ exists and equals $k \int_a^b f \, d\varphi$, where k is any constant;

(II) If $\int_a^b f \, d\varphi$ exists and (α, β) is any subinterval of (a, b) , then $\int_\alpha^\beta f \, d\varphi$ exists, and

$$\int_a^b f \, d\varphi = \int_a^c f \, d\varphi + \int_c^b f \, d\varphi, \quad (a < c < b);$$

(III) Let $f = f_1 + f_2$ and $\varphi = \varphi_1 + \varphi_2$. If $\int_a^b f_i \, d\varphi_j$ exists ($i, j=1, 2$), then $\int_a^b f \, d\varphi$ exists, and

$$\int_a^b f \, d\varphi = \sum_{i,j=1}^2 \int_a^b f_i \, d\varphi_j.$$

The integration by parts theorem also holds for these four integrals; i.e.,

(IV) If $\int_a^b f \, d\varphi$ exists, then $\int_a^b \varphi \, df$ exists, and

$$\int_a^b f \, d\varphi + \int_a^b \varphi \, df = [f\varphi]_a^b.$$

The proof of this theorem for the mean integrals is likewise based on the definitions, and is obtained immediately from the identity,

$$\sum_{i=1}^n M_i \varphi \Delta_i f = [f\varphi]_a^b - \sum_{i=1}^n M_i f \Delta_i \varphi,$$

but the proof for the case of the S-integrals is more involved.¹

The question of the existence of these four integrals has been considered in detail by H. L. Smith. Three of his conditions are applied frequently in this paper; they will be stated here for convenience in reference. The condition which is most easily applied is expressed in terms of the oscillation of the approximating sums $S(f\Delta\varphi)$ and $S_m(f\Delta\varphi)$. The oscillation function $\omega[S(f\Delta\varphi); c, d]$ is defined as the least upper bound of the absolute values of the difference of the values of $S(f\Delta\varphi)$ for any two subdivisions of (c, d) ; i.e.,

$$\omega[S(f\Delta\varphi); c, d] = \text{LUB} \left[\left| \sum_{\sigma_1} f(\xi_1) \Delta_1 \varphi - \sum_{\sigma_2} f(\xi_1) \Delta_1 \varphi \right| \right]$$

for all subdivisions σ_1, σ_2 of (c, d) .

1. This theorem was first proved for the FS-integral by B. Dushnik (Du); for a proof for the NS-integral, see e.g., Ho, p. 539.

The oscillation of $S(f\Delta\phi)$ over a subdivision σ of the fundamental interval (a,b) is then given by the sum of the oscillations over the subintervals of the subdivision; thus,

$$\omega_{\sigma} S(f\Delta\phi) = \sum_{i=1}^n \omega \left[S(f\Delta\phi); x_{i-1}, x_i \right].$$

When no confusion can arise, the oscillation over a subinterval $\Delta_i x$ will be denoted by $\omega_i S(f\Delta\phi)$, so that the last relation may be written in the simpler form

$$\omega_{\sigma} S(f\Delta\phi) = \sum_{i=1}^n \omega_i S(f\Delta\phi).$$

Similarly, the oscillation of $S_m(f\Delta\phi)$ over $\Delta_i x$ and over (a,b) will be denoted by $\omega_i S_m(f\Delta\phi)$ and $\omega_{\sigma} S_m(f\Delta\phi)$, respectively. It will be convenient to introduce the additional symbols $V\phi(x)$ and $\Delta_i V\phi$ to denote the total variation of $\phi(x)$ over (a,x) , $(a < x \leq b)$, and over (x_{i-1}, x_i) , respectively.

The existence conditions may then be stated as follows:¹

THEOREM A. A necessary and sufficient condition that $NS \int_a^b f \, d\phi$ or $NS_m \int_a^b f \, d\phi$ exist is that

$$\lim_N \omega_{\sigma} S(f\Delta\phi) = 0, \quad \lim_N \omega_{\sigma} S_m(f\Delta\phi) = 0,$$

respectively, and similarly for the F-integrals.

THEOREM B. A necessary condition that $NS \int_a^b f \, d\phi$ exist is that

$$\lim_N \sum_{i=1}^n \omega_i f \left| \Delta_i \phi \right| = 0,$$

and similarly for the FS-integral.

L. Sm, pp. 493-506. Theorem A is Smith's Theorem 1, Theorem B is part of his Theorem N₁, and Theorem C is contained in Corollary 2 of his general existence theorem (p. 506). The condition of Theorem B is stated to be necessary for the Sm-integrals also, but the example given in §7 below shows that this is not the case. Consequently Smith's Theorem N₂, which is derived from this result, also applies only to the S-integrals. The part of the theorem concerning the Sm-integrals is disproved by Theorem 3, §5, below.

THEOREM C. If $\phi(x)$ is of bounded variation, a necessary and sufficient condition that $NS \int_a^b f d\phi$ exist is that

$$\lim_N \sum_{i=1}^n \omega_i f \Delta_i V\phi = 0;$$

and similarly for the FS-integral.

It follows as a corollary to Theorem C that $\int_a^b f d\phi$ exists if $f(x)$ is continuous and $\phi(x)$ is of bounded variation in (a,b) .

CHAPTER II

EXISTENCE OF THE MEAN INTEGRALS

4. Some necessary conditions. A necessary condition for the existence of the NSm-integral, corresponding to the condition that $f(x)$ and $\phi(x)$ have no common discontinuities when the integral $\text{NS} \int_a^b f \, d\phi$ exists, is as follows.

THEOREM 1. If $\text{NSm} \int_a^b f \, d\phi$ exists, then for any point x in (a, b) ,

$$(4.1) \quad D(f, \phi; x) = \lim_{\substack{\varepsilon \rightarrow +0 \\ \delta \rightarrow +0}} \begin{vmatrix} 1 & f(x) & \phi(x) \\ 1 & f(x - \varepsilon) & \phi(x - \varepsilon) \\ 1 & f(x + \delta) & \phi(x + \delta) \end{vmatrix} = 0.$$

Proof. Let σ_1 be a subdivision of (a, b) with $x - \varepsilon$ and $x + \delta$ as consecutive points, and let σ_2 be the subdivision formed from σ_1 by the insertion of the point x . Then the approximating sums corresponding to σ_1 and σ_2 are, respectively,

$$\begin{aligned} \lambda_1 &= \sum_0 M_i f \Delta_i \phi + \frac{1}{2} [f(x - \varepsilon) + f(x + \delta)] [\phi(x + \delta) - \phi(x - \varepsilon)], \\ \lambda_2 &= \sum_0 M_i f \Delta_i \phi + \frac{1}{2} \left\{ [f(x - \varepsilon) + f(x)] [\phi(x) - \phi(x - \varepsilon)] + \right. \\ &\quad \left. [f(x) + f(x + \delta)] [\phi(x + \delta) - \phi(x)] \right\}, \end{aligned}$$

where $\sum_0$ extends over all the subintervals of σ_1 except $(x - \varepsilon, x + \delta)$. After subtracting and rearranging terms, we find that

$$\lim_N (\lambda_1 - \lambda_2) = \frac{1}{2} D(f, \phi; x).$$

Since $\lim_N \lambda_1$ and $\lim_N \lambda_2$ must be equal if the integral is to exist, it follows that $D(f, \phi; x) = 0$.

Another necessary condition for the existence of the mean integrals is given by

THEOREM 2. If $\text{Sm} \int_a^b f \, d\phi$ exists, then

$$(4.2) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \begin{vmatrix} 1 & f(\xi_i) & \phi(\xi_i) \\ 1 & f(x_{i-1}) & \phi(x_{i-1}) \\ 1 & f(x_i) & \phi(x_i) \end{vmatrix} = 0, \quad (x_{i-1} \leq \xi_i \leq x_i),$$

where the limit is taken in the same sense as in the integral.

The proof will be given for the norm limit. Let σ_0 be a subdivision of (a, b) whose norm does not exceed the N_e of (1.3), which is now assumed to be written for the NSm-integral rather than the NS-integral. Then if σ is a subdivision formed from σ_0 by inserting a point ξ_i in each subinterval (x_{i-1}, x_i) ,

$$\begin{aligned} \sum_{\sigma} M_i f \Delta_i \varphi - \sum_{\sigma_0} M_i f \Delta_i \varphi &= \sum_{i=1}^n \frac{1}{2} \left\{ [f(x_{i-1}) + f(\xi_i)] [\varphi(\xi_i) - \varphi(x_{i-1})] + [f(\xi_i) + f(x_i)] [\varphi(x_i) - \varphi(\xi_i)] \right. \\ &\quad \left. - [f(x_{i-1}) + f(x_i)] [\varphi(x_i) - \varphi(x_{i-1})] \right\} \\ &= \sum_{i=1}^n -\frac{1}{2} \begin{vmatrix} 1 & f(\xi_i) & \varphi(\xi_i) \\ 1 & f(x_{i-1}) & \varphi(x_{i-1}) \\ 1 & f(x_i) & \varphi(x_i) \end{vmatrix} \end{aligned}$$

But $\lim_N \left| \sum_{\sigma} M_i f \Delta_i \varphi - \sum_{\sigma_0} M_i f \Delta_i \varphi \right| = 0$ since $\text{NSm} \int_a^b f d\varphi$

exists, and therefore (4.2) holds in the N sense.

The condition of the theorem may evidently be extended to allow the insertion of any number of additional points in each subinterval, and as thus extended the condition is also sufficient for the existence of the FSm-integral. Thus we have the following corollaries.

COROLLARY 1. A necessary and sufficient condition that FSm $\int_a^b f d\varphi$ exist is that

$$(4.3) \quad \lim_F \sum_{i=1}^n \sum_{j=1}^{m_i} \begin{vmatrix} 1 & f(x_{k_{ij}-1}) & \varphi(x_{k_{ij}-1}) \\ 1 & f(x_{k_{ij}}) & \varphi(x_{k_{ij}}) \\ 1 & f(x_i) & \varphi(x_i) \end{vmatrix} = 0,$$

where $x_{k_0} = x_{i-1}$, and the x_k 's are arbitrary points in (x_{i-1}, x_i) .

COROLLARY 2. The conditions (4.1) and (4.3) together are necessary and sufficient for the existence of the integral $\text{NSm} \int_a^b f d\varphi$.

The sufficiency of the condition in Corollary 1 follows from the fact that the sum in (4.3) may be reduced to

$$(4.4) \quad \sum_{\sigma} M_i f \Delta_i \varphi - \sum_{\sigma_0} M_i f \Delta_i \varphi,$$

where σ_0 is a fixed subdivision of (a, b) and $\sigma > \sigma_0$. If σ_0 is finer than the σ_e of (1.5), which is now assumed to

be written for the FSm-integral rather than the FS-integral, then (4.3) implies that the F limit of (4.4) is zero, and hence that $\text{FSm} \int_a^b f \, d\varphi$ exists. Corollary 2 is obtained from a theorem of B. C. Getchell¹ on functions of intervals, which includes the special case that the integral $\text{NSm} \int_a^b f \, d\varphi$ exists if $\text{FSm} \int_a^b f \, d\varphi$ exists and $D(f, \varphi; x) = 0, (a \leq x \leq b)$, and conversely.

The condition (4.3) is equivalent to the following theorem of H. L. Smith² on the existence of the mean integrals.

A necessary condition that $\text{Sm} \int_a^b f \, d\varphi$ exist is that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta_i \alpha(f, \varphi) = 0,$$

where $\Delta_i \alpha(f, \varphi)$ is the area of the convex polygon enclosing the points whose coordinates are $(f(x), \varphi(x))$ when $x_{i-1} \leq x \leq x_i$, and where the limit is taken in the same sense as in the integral.

The derivation of condition (4.3) is much more direct than Smith's proof of this theorem, and the condition is in a form which is simpler and easier to apply than Smith's condition. Smith shows that his condition is sufficient for the existence of the mean integrals if it is assumed in addition that $\varphi(x)$ is related in a certain way to $f(x)$, and in particular if $\varphi(x)$ is monotonic, but in contrast to this, the condition (4.3) is itself sufficient for the existence of the FSm-integral.

5. A sufficient condition for the existence of the mean integrals. An analogue for the mean integrals of the condition that $\text{S} \int_a^b f \, d\varphi$ exists when $f(x)$ is continuous and $\varphi(x)$ is of bounded variation is as follows.

THEOREM 3. If $f(x)$ is bounded and has discontinuities of the first kind only, and if $\varphi(x)$ is of bounded variation, then $\text{FSm} \int_a^b f \, d\varphi$ exists; if in addition, $D(f, \varphi; x) = 0$ at every point x of (a, b) , then $\text{NSm} \int_a^b f \, d\varphi$ exists.

Place $B\varphi(\xi) = \varphi(\xi + 0) - \varphi(\xi - 0)$, and let the discontinuities of $\varphi(x)$ be denoted by $\xi_1, \xi_2, \xi_3, \dots$, and let them be arranged so that $|B\varphi(\xi_k)| \geq |B\varphi(\xi_{k+1})|$ for every k .

1. Dissertation, University of Michigan, 1934.

2. Sm, Theorem N3, p. 498.

Now place

$$(5.1) \quad \varphi_m(x) = \psi(x) + \sum_{k=1}^m \theta_k(x),$$

$$(5.2) \quad \varphi(x) = \psi(x) + \sum_{k=1}^{\infty} \theta_k(x),$$

where $\psi(x)$ is continuous, and $\theta_k(x)$ is defined by

$$(5.3) \quad \begin{cases} \theta_k(x) = 0 & \text{when } x < \xi_k, \\ = \varphi(\xi_k) - \varphi(\xi_k - 0) & x = \xi_k, \\ = B\varphi(\xi_k) & x > \xi_k. \end{cases}$$

The theorem will be proved by showing that $\text{Sm} \int_a^b f \, d\varphi$ exists, after which the existence of $\text{Sm} \int_a^b f \, d\varphi$ will be obtained by letting m become infinite. The proof is made to depend on the following lemmas, in the last of which the symbol $V_t u$ is introduced to denote the total variation of $u(x)$ from a to b .

LEMMA 1. If $f(x)$ is bounded and has only discontinuities of the first kind, and if $\psi(x)$ is continuous and of bounded variation in (a, b) , then $\text{NS} \int_a^b f \, d\psi$ exists,¹ and hence the NSm - and FSm -integrals exist.

LEMMA 2. If $f(x)$ is bounded and has only discontinuities of the first kind, and if $\theta = \theta_k$ is as defined in (5.3), then $\text{FSm} \int_a^b f \, d\theta$ exists; if in addition $D(f, \theta; \xi) = 0$, then $\text{NSm} \int_a^b f \, d\theta$ exists.

LEMMA 3. If $\text{Sm} \int_a^b f \, d\varphi_m$ exists for every m , and if $\lim_{n \rightarrow \infty} V_t(\varphi - \varphi_m) = 0$, then $\text{Sm} \int_a^b f \, d\varphi$ exists, and

$$\lim_{n \rightarrow \infty} \text{Sm} \int_a^b f \, d\varphi_m = \text{Sm} \int_a^b f \, d\varphi.$$

In Lemma 2, let ξ be a point of a subdivision of (a, b) ; then for any finer subdivision with $\xi - \delta$, ξ , $\xi + \delta'$ as consecutive points, we will have

$$\begin{aligned} \sum_{i=1}^m M_i f \Delta_i \theta &= \frac{1}{2} [f(\xi - \delta) + f(\xi)] [\theta(\xi) - \theta(\xi - \delta)] + \\ &\quad \frac{1}{2} [f(\xi) + f(\xi + \delta')] [\theta(\xi + \delta') - \theta(\xi)]. \end{aligned}$$

1. See Ho, p. 545.

Passing to the limit, and making use of (5.3), we get

$$\begin{aligned} \lim_F \sum_{i=1}^n M_i f \Delta_i \theta &= \frac{1}{2} \left[f(\xi - 0) + f(\xi) \right] \left[\varphi(\xi) - \varphi(\xi - 0) \right] + \\ &\quad \frac{1}{2} \left[f(\xi) + f(\xi + 0) \right] \left[\varphi(\xi + 0) - \varphi(\xi) \right] \\ &= M f(\xi) B\varphi(\xi) - \frac{1}{2} D(f, \varphi; \xi), \end{aligned}$$

where $M f(\xi) = \frac{1}{2} [f(\xi - 0) + f(\xi + 0)]$. Both terms on the right exist from the restrictions on $f(x)$ and $\varphi(x)$. Hence $F S m \int_a^b f d\varphi$ exists.

According as ξ is or is not a point of subdivision of (a, b) , the approximating sum

$$\lambda = \lim_N \sum_{i=1}^n M_i f \Delta_i \varphi$$

has the value

$$\lambda_1 = -\frac{1}{2} D(f, \theta; \xi) + M f(\xi) B\theta(\xi) \quad \text{or} \quad \lambda_2 = M f(\xi) B\theta(\xi).$$

Since $D(f, \theta; \xi) = D(f, \varphi; \xi) = 0$, then in either case

$$\lambda = M f(\xi) B\varphi(\xi),$$

so that $N S m \int_a^b f d\theta$ exists. This completes the proof of Lemma 2.

To prove Lemma 3, let the same subdivision of (a, b) be used to form the approximating sums for $S m \int_a^b f d\varphi$ and $S m \int_a^b f d\varphi_m$. Then if L denotes the least upper bound of $|f(x)|$ on (a, b) ,

$$\begin{aligned} \left| \sum_{i=1}^n M_i f \Delta_i \varphi - \sum_{i=1}^n M_i f \Delta_i \varphi_m \right| &\leq L \sum_{i=1}^m |\Delta_i \varphi - \Delta_i \varphi_m| \\ &\leq L \sum_{i=1}^n V_i (\varphi - \varphi_m) \leq L \cdot V_t (\varphi - \varphi_m). \end{aligned}$$

Passing to the limit, we get

$$\lim_{m \rightarrow \infty} \sum_{i=1}^n M_i f \Delta_i \varphi_m = \sum_{i=1}^n M_i f \Delta_i \varphi,$$

and the limit exists uniformly in N (or F). The proof of the lemma is completed in the same way as the proof for the interchange of order in iterated limits in which one inner limit exists uniformly (see H_1 , pp. 290-291), and will therefore be omitted.

Returning to the proof of the theorem, we see from Lemmas 1 and 2 and from (III), §3, that $S m \int_a^b f_m d\varphi$ exists, and that

$$\text{Sm} \int_a^b f \, d\varphi_m = \text{Sm} \int_a^b f \, d\psi + \sum_{k=1}^m \text{Sm} \int_a^b f \, d\theta_k,$$

where the integral is taken in either the N or F sense.

Subtracting (5.1) from (5.2), we get

$$\begin{aligned} \varphi(x) - \varphi_m(x) &= \sum_{k=m+1}^{\infty} \theta_k(x). \\ \therefore V_t(\varphi - \varphi_m) &= \sum_{k=m+1}^{\infty} |B\varphi(\xi_k)|. \end{aligned}$$

Since $\varphi(x)$ is of bounded variation, the right member of the last equality can be made arbitrarily small by taking m sufficiently large, so that

$$\lim_{m \rightarrow \infty} V_t(\varphi - \varphi_m) = 0.$$

Thus the hypotheses of Lemma 3 are satisfied. It follows at once that $\text{Sm} \int_a^b f \, d\varphi$ exists, and the theorem is completely established.

As an immediate consequence of the theorem, we have the

COROLLARY. The integral $\text{FSm} \int_a^b f \, d\varphi$ exists if both $f(x)$ and $\varphi(x)$ are of bounded variation; and $\text{NSm} \int_a^b f \, d\varphi$ exists if both $f(x)$ and $\varphi(x)$ are regular functions of bounded variation, i.e.,

$$f(x) = M f(x), \quad \varphi(x) = M \varphi(x), \quad (a \leq x \leq b).$$

The theorem shows that the class of functions which have at most discontinuities of the first kind plays the same role in the theory of the existence of the mean integrals as the class of continuous functions plays in the case of the ordinary integrals. However, the analogy between the S- and the Sm-integrals for continuous functions and functions having discontinuities of the first kind, respectively, is not complete. Thus the following properties on the convergence of sequences of integrals do not hold for the mean integrals if $f(x)$ is allowed to have discontinuities of the first kind instead of being continuous.

(1) If $\{f_n(x)\}$ is a sequence of functions which are integrable with respect to $\varphi(x)$ and such that $f_n(x) \rightarrow f(x)$ monotonically on (a, b) , then $\int_a^b f_n \, d\varphi \rightarrow \int_a^b f \, d\varphi$.

(2) If $f(x)$ is continuous and $\{\varphi_n(x)\}$ is a sequence of functions of uniformly bounded variation such that $\varphi_n(x) \rightarrow \varphi(x)$ at a set of points which includes a and b and

is dense on (a, b) , then $\int_a^b f \, d\varphi$ exists, and¹

$$\lim_{n \rightarrow \infty} \int_a^b f \, d\varphi_n = \int_a^b f \, d\varphi.$$

These properties are disproved for the mean integrals by means of examples. Thus in (1), take $f(x) = 1$ or 0 according as $0 \leq x \leq 1/n$ or $1/n < x \leq 1$, and take $f_0(0) = 1$, $f_0(x) = 0$ elsewhere in $(0, 1)$, and let $\varphi(x)$ be a function of bounded variation which is discontinuous on the right at $x = 0$. Then $f_1 \geq f_2 \geq f_3 \geq \dots \rightarrow f_0$, and

$$Sm \int_0^1 f_0 \, d\varphi = \frac{1}{2} [\varphi(0 + 0) - \varphi(0)],$$

$$Sm \int_0^1 f_n \, d\varphi = \frac{1}{2} \left[\varphi\left(\frac{1}{n}\right) + \varphi\left(\frac{1}{n} + 0\right) \right] - \varphi(0)$$

$$\rightarrow \varphi(0 + 0) - \varphi(0),$$

so that $Sm \int f_n \, d\varphi$ does not converge to $Sm \int f_0 \, d\varphi$. In (2), take $f(0) = \varphi(0) = 1$, and $f(x) = \varphi(x) = 0$ elsewhere in $(0, 1)$, and let $\varphi_n(x) = 1$ or 0 according as $0 \leq x \leq 1/n$ or $1/n < x \leq 1$. Then $Sm \int_0^1 f \, d\varphi = -\frac{1}{2}$, but $Sm \int_0^1 f \, d\varphi_n = 0$ for every n , so that $\lim Sm \int_0^1 f \, d\varphi_n \neq Sm \int_0^1 f \, d\varphi$.

1. Du, p. 39; see also Bray (B, Theorem 3, p. 180). The proofs given are for the S-integrals, but the theorem is evidently valid for the Sm-integral also when $f(x)$ is continuous.

CHAPTER III

GENERAL PROPERTIES OF THE INTEGRALS

6. Additional elementary properties. The properties stated in §3 were obtained without restricting in advance the class of functions for which the integrals are defined. Additional properties, analogous to those of the Riemann integral, may be derived by requiring that the integrator be monotonic or of bounded variation. The following are easily established for all four types of integrals.¹

(V) If $l \leq f(x) \leq L$ and $\varphi(x)$ is monotonic non-decreasing on (a, b) , then if $\int_a^b f \, d\varphi$ exists,

$$l [\varphi(b) - \varphi(a)] \leq \int_a^b f \, d\varphi \leq L [\varphi(b) - \varphi(a)].$$

(VI) If $f_1(x) \geq f_2(x)$ and $\varphi(x)$ is monotonic non-decreasing on (a, b) , then if the integrals exist,

$$\int_a^b f_1 \, d\varphi \geq \int_a^b f_2 \, d\varphi.$$

(VII). Suppose $\int_a^b f \, d\varphi$ exists, and place

$$F(x) = \int_a^x f \, d\varphi, \quad (a \leq x \leq b).$$

If $\varphi(x)$ is of bounded variation, then $F(x)$ is of bounded variation, and is continuous at all points where $\varphi(x)$ is continuous.

The analogues of the theorems on absolute integrability and on the integrability of the product of two functions hold for the S-integrals when the integrator is monotonic or of bounded variation, respectively. Thus

(VIII) If $\int_a^b f \, d\varphi$ and $\int_a^b g \, d\varphi$ exist, where $f(x)$ and $g(x)$ are bounded and $\varphi(x)$ is of bounded variation on (a, b) , then $\int_a^b f g \, d\varphi$ exists.

(IX) If $\int_a^b f \, d\varphi$ exists and if $\varphi(x)$ is monotonic non-decreasing on (a, b) , then $\int_a^b |f| \, d\varphi$ exists, and

$$\left| \int_a^b f \, d\varphi \right| \leq \int_a^b |f| \, d\varphi.$$

1. The proofs of (V) and (VI) follow directly from the definitions of the integrals. A proof of (VII) is given by Lebesgue (L, p. 255) for the case of the NS-integral, but it is valid for the other types as well.

The following property is frequently more convenient to apply than (IX).

(X) If $S \int_a^b f \, d\varphi$ exists and if $\varphi(x)$ is of bounded variation, then $S \int_a^b |f| \, dV\varphi$ exists, and

$$\left| S \int_a^b f \, d\varphi \right| \leq S \int_a^b |f| \, dV\varphi.$$

In VIII), let L denote the greatest value of $|f(x)|$ and $|g(x)|$ on (a, b) ; then

$$\omega_1(fg) \leq L\omega_1 f + L\omega_1 g.$$

$$\therefore \sum_{i=1}^n \omega_1(fg) |\Delta_i \varphi| \leq L \sum_{i=1}^n (\omega_1 f |\Delta_i \varphi| + \omega_1 g |\Delta_i \varphi|).$$

Since this inequality holds for any subdivision of (a, b) , and since the limit of the right member is zero by Theorem C, then

$$\lim_{i=1}^n \omega_1(fg) |\Delta_i \varphi| = 0.$$

Hence $S \int_a^b f g \, d\varphi$ exists by Theorem C.

This property does not hold for the Sm -integrals if $\varphi(x)$ is not of bounded variation, nor for the NSm -integral even when $\varphi(x)$ is of bounded variation. For if $f(0) = \varphi(0) = 0$, and $f(x) = \varphi(x) = \sin 1/x$ ($0 < x \leq 1$), then $Sm \int_0^1 f \, d\varphi$ exists, but $Sm \int_0^1 f^2 \, d\varphi$ does not exist. To show this, let σ_0 be any subdivision of $(0, 1)$ with $0, x_1$ as consecutive points, and suppose $f(x_1) = h$. If $h \leq 0$, take $\sigma = \sigma_0 + x' + x''$, where $0 < x' < x'' < x_1$ and $f(x') = 1, f(x'') = -1$. Then

$$\sum_{\sigma} M_i |f| \Delta_i f - \sum_{\sigma_0} M_i |f| \Delta_i f = -1 - h^2/2 \leq -1.$$

If $h \geq 0$, take x' and x'' so that $f(x') = -1, f(x'') = 1$; then this difference will be ≥ 1 . Therefore $|\Sigma_{\sigma} - \Sigma_{\sigma_0}| \geq 1$, and since such a subdivision σ can be found for any subdivision σ_0 , it follows that $\lim_F \Sigma_{\sigma}$ does not exist. Now let $f(x) = -1, 1/2, 2$ according as $0 \leq x < 1/2, x = 1/2, 1/2 < x \leq 1$; then $D(f, f; 1/2) = 0$ and $D(f^2, f; 1/2) = -27/4 \neq 0$. Hence by Theorems 1 and 3, $NSm \int_0^1 f \, d\varphi$ exists, but $Nsm \int_0^1 f^2 d\varphi$ does not exist.

To prove (IX), we have $\omega_1 |f| \leq \omega_1 f$, from which we get for any subdivision of (a, b) ,

$$(6.1) \quad \sum_{i=1}^n \omega_i |f| \Delta_i V\varphi \leq \sum_{i=1}^n \omega_i f \Delta_i V\varphi.$$

Since $\varphi(x)$ is monotonic, and since $S \int_a^b f \, d\varphi$ exists by hypothesis, then by Theorem C the limit (in either the N or the F sense) of the right member of (6.1) is zero. Consequently the limit of the left member is zero, and hence $S \int_a^b |f| \, d\varphi$ exists. Now by (VI),

$$\begin{aligned} -S \int_a^b |f| \, d\varphi &\leq S \int_a^b f \, d\varphi \leq S \int_a^b |f| \, d\varphi. \\ \therefore \left| S \int_a^b f \, d\varphi \right| &\leq S \int_a^b |f| \, d\varphi. \end{aligned}$$

The above proof is valid only for the S-integrals, for the condition $\lim \sum \omega_i f \Delta_i V\varphi = 0$ is not necessary for the existence of $S \int_a^b f \, d\varphi$. In fact, the theorem is not true for the NSm-integral, as may be shown by applying Theorems 1 and 3. Thus if $f(x) = -1, 1/2, 2$, and $\varphi(x) = -1, 0, 1$ according as $0 \leq x < 1/2$, $x = 1/2$, $1/2 < x \leq 1$, respectively, then $D(f, \varphi; 1/2) = 0$ so that $NSm \int_0^1 f \, d\varphi$ exists, but $D(|f|, \varphi; 1/2) \neq 0$, so that $NSm \int_0^1 |f| \, d\varphi$ does not exist. However, it still follows from (VI) that

$$\left| Sm \int_a^b f \, d\varphi \right| \leq Sm \int_a^b |f| \, d\varphi$$

if $Sm \int_a^b |f| \, d\varphi$ exists.

Property (X) is easily derived from the fact that the existence of $S \int_a^b f \, d\varphi$ is equivalent to the existence of $S \int_a^b f \, dV\varphi$ when $\varphi(x)$ is of bounded variation (Hy₂, p. 237). Since $V\varphi(x)$ is monotonic non-decreasing, it follows from (IX) that $S \int_a^b |f| \, dV\varphi$ exists. Hence by passing to the limit in the inequality

$$\left| \sum_0 f(\xi_i) \Delta_i \varphi \right| \leq \sum_0 |f(\xi_i)| \Delta_i V\varphi,$$

which is valid for any subdivision σ of (a, b) , we get (X).

Here again the theorem fails to hold for the NSm-integrals. For if $\varphi(x) = 0, 1, -1$, according as $0 \leq x < 1/2$, $x = 1/2$, $1/2 < x \leq 1$, then $NSm \int_0^1 \varphi \, d\varphi$ exists, but $D(\varphi, V\varphi; 1/2) \neq 0$ and $D(|\varphi|, V\varphi; 1/2) \neq 0$, so that neither $NSm \int_0^1 \varphi \, dV\varphi$ nor $NSm \int_0^1 |\varphi| \, dV\varphi$ exists.

7. The substitution theorem. Another important property of the S-integrals is the following, which is known as the substitution theorem.¹

1. Proved by J. Hyslop (Hy₁) for the case of the NS-integral. The theorem may be proved directly for both the NS- and FS-integrals by making use of Theorem A.

(XI) If $h(x)$ is bounded in (a, b) and if $S \int_a^b f d\varphi$ exists, then $F(x) = S \int_a^x f d\varphi$ exists ($a \leq x \leq b$), and

$$S \int_a^b h dF = S \int_a^b h f d\varphi,$$

whenever either integral exists.

This property does not hold for the Sm-integrals, however, as a simple example will show. Thus if $f(x) = h(x) = \varphi(x) = 0$ or 1 according as $0 \leq x < 1/2$ or $1/2 \leq x \leq 1$, the conditions of the theorem are satisfied in both the N and F senses, but $Sm \int_0^1 h dF = 1/4$, while $Sm \int_0^1 h f d\varphi = 1/2$. Since the difficulty seems to arise from the common discontinuities of the functions, it is natural to impose further restrictions on these functions in order to obtain a corresponding theorem for the Sm-integrals. Such results may be derived from the following general theorem, which is valid in both the N and F senses.

THEOREM 4. Suppose $Sm \int_a^b f d\varphi$ exists, and place

$$F(x) = Sm \int_a^x f d\varphi, \quad (a \leq x \leq b).$$

Then if $h(x)$ is bounded on (a, b) , a necessary and sufficient condition that $Sm \int_a^b h f d\varphi$ and $Sm \int_a^b h dF$ exist simultaneously and be equal is that

$$(7.1) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta_i f \Delta_i h \Delta_i \varphi = 0,$$

the limit being taken in the same sense as in the integral.

Using the same subdivision of (a, b) in each case, form the approximating sums for the two integrals, and place

$$(7.2) \quad R = \sum_{i=1}^n M_i h \Delta_i F - \sum_{i=1}^n M_i (fh) \Delta_i \varphi.$$

Letting u_i denote $u(x_i)$, we may write

$$M_i(fh) = \frac{1}{4}(f_i + f_{i-1})(h_i + h_{i-1}) + \frac{1}{4}(f_i - f_{i-1})(h_i - h_{i-1}).$$

$$(7.3) \quad \therefore M_i(fh) = M_i f M_i h + \frac{1}{4} \Delta_i f \Delta_i h.$$

Substituting from (7.3) into (7.2), we get

$$(7.4) \quad R = \sum_{i=1}^n M_i h (\Delta_i F - M_i f \Delta_i \varphi) - \frac{1}{4} \sum_{i=1}^n \Delta_i f \Delta_i h \Delta_i \varphi.$$

$$\therefore |R| \leq L \sum_{i=1}^n \left| \Delta_i F - M_i f \Delta_i \varphi \right| + \frac{1}{4} \left| \sum_{i=1}^n \Delta_i f \Delta_i h \Delta_i \varphi \right|$$

$$\leq L \omega_{Sm}(f, \Delta \varphi) + \frac{1}{4} \left| \sum_{i=1}^n \Delta_i f \Delta_i h \Delta_i \varphi \right|,$$

where L denotes the maximum of $|h(x)|$ on (a, b) .

The sufficiency of the condition follows from this inequality since $\lim \omega_\sigma \text{Sm}(f\Delta\phi) = 0$ from the existence of $\text{Sm} \int_a^b f d\phi$. To show that the condition is also necessary, write (7.4) in the form

$$\frac{1}{4} \sum_{i=1}^n \Delta_i f \Delta_i h \Delta_i \phi = \sum_{i=1}^n M_i h (\Delta_i F - M_i f \Delta_i \phi) - R.$$

Then

$$\frac{1}{4} \left| \sum_{i=1}^n \Delta_i f \Delta_i h \Delta_i \phi \right| \leq L \omega_\sigma \text{Sm}(f\Delta\phi) + |R|,$$

from which (7.1) follows, since $\lim R = 0$ by hypothesis, and $\lim \omega_\sigma \text{Sm}(f\Delta\phi) = 0$ as just noted.

COROLLARY 1. If $\text{Sm} \int_a^b f d\phi$ exists, if $\phi(x)$ is of bounded variation, and if $f(x)$ and $h(x)$ are bounded functions with no common discontinuities, then

$$\text{Sm} \int_a^b h dF = \text{Sm} \int_a^b h f d\phi$$

whenever either integral exists.

The proof of the corollary consists in showing that condition (7.1) is satisfied under the given restrictions on the functions. By hypothesis, for every $\epsilon > 0$ there exists a subinterval Δx about each point x of (a, b) such that at least one of the relations $|\Delta f| < \epsilon$ or $|\Delta h| < \epsilon$ is satisfied. Hence by the Borel Theorem, there exists a finite number n_ϵ of intervals (x_j, x_k) which cover (a, b) for which this is true. Let $N = \min. |x_j - x_k|$, $(j, k=1, 2, \dots, n_\epsilon; j \neq k)$. Then if σ is a subdivision of (a, b) such that $\Delta_i x < N$, $(i=1, 2, \dots, n)$, each $\Delta_i x$ will lie inside one of the n_ϵ subintervals, and hence at least one of the relations $|\Delta_i f| < \epsilon$ or $|\Delta_i h| < \epsilon$, $(i=1, 2, \dots, n)$, will be satisfied. Therefore, if L denotes the maximum of $|f(x)|$, $|h(x)|$, on (a, b) , we will have

$$\left| \sum_{i=1}^n \Delta_i f \Delta_i h \Delta_i \phi \right| \leq \epsilon \cdot 2L \sum_{i=1}^n |\Delta_i \phi| \leq \epsilon \cdot 2L \cdot V_\phi(b).$$

Therefore $\lim_N \sum \Delta_i f \Delta_i h \Delta_i \phi = 0$, and consequently $\lim_F \sum \Delta_i f \Delta_i h \Delta_i \phi = 0$, which proves the corollary.

COROLLARY 2. Let $h(x)$ and $\phi(x)$ be of bounded variation, and $f(x)$ bounded on (a, b) ; assume, moreover, that these functions are not all discontinuous simultaneously. Then if $\text{Sm} \int_a^b f d\phi$ exists,

$$\text{Sm} \int_a^b h f d\varphi = \text{Sm} \int_a^b h dF$$

whenever either integral exists.

As in Corollary 1, it is only necessary to show that condition (7.1) is satisfied. To do this, place

$$(7.5) \quad \sum_0 \Delta_1 f \Delta_1 h \Delta_1 \varphi = \sum_1 \Delta_1 f \Delta_1 h \Delta_1 \varphi + \sum_2 \Delta_1 f \Delta_1 h \Delta_1 \varphi,$$

where $\sum_1$ extends over the subintervals of σ in which $\varphi(x)$ is continuous, and $\sum_2$ is over the rest. Since either $f(x)$ or $h(x)$ is continuous where $\varphi(x)$ is discontinuous, then for every $\epsilon > 0$ there is an N_ϵ such that if $N < N_\epsilon$,

$$(7.6) \quad \left| \sum_2 \Delta_1 f \Delta_1 h \Delta_1 \varphi \right| \leq 2L \cdot \epsilon \cdot \sum_2 |\Delta \varphi| \leq 2L V \varphi(b) \cdot \epsilon$$

where L denotes the maximum of $|f(x)|$, $|h(x)|$, $|\varphi(x)|$ on (a, b) . It may be assumed that the N_ϵ has been taken so that $|\Delta \varphi| < \epsilon$ in $\sum_1$, in which case

$$(7.7) \quad \left| \sum_1 \Delta_1 f \Delta_1 h \Delta_1 \varphi \right| \leq 2L \epsilon \sum_1 |\Delta_1 h| \leq 2L V h(b) \cdot \epsilon.$$

It follows from (7.5), (7.6), and (7.7) that (7.1) holds in the N sense, and hence also in the F sense.

COROLLARY 3. If $\text{Sm} \int_a^b f d\varphi$ exists, if $f(x)$ is continuous, and if $\sum_0 |\Delta_1 h| |\Delta_1 \varphi|$ is bounded for all subdivisions of (a, b) , then

$$\text{Sm} \int_a^b h dF = \text{Sm} \int_a^b h f d\varphi$$

whenever either integral exists.

From the continuity of $f(x)$, for every $\epsilon > 0$ there exists an N_ϵ such that $|\Delta_1 f| < \epsilon$ when $\Delta_1 x < N_\epsilon$. Therefore

$$\left| \sum_{i=1}^n \Delta_1 f \Delta_1 h \Delta_1 \varphi \right| < \epsilon \sum_{i=1}^n |\Delta_1 h \Delta_1 \varphi|, \quad (\Delta_1 x < N_\epsilon).$$

Since the sum on the right is bounded, it follows that the limit of the left member is zero in the N sense, and hence also in the F sense.

As a special case of the above, we have

COROLLARY 4. If $f(x)$ is continuous and $h(x)$ and $\varphi(x)$ are of bounded variation on (a, b) , then

$$\text{Sm} \int_a^b h dF = \text{Sm} \int_a^b h f d\varphi$$

whenever either integral exists.

8. Properties derived from the substitution theorem.

The substitution and the integration by parts theorems, together with other elementary properties of the integrals, may be used to derive an extended integration by parts theorem and a theorem on the interchange of order in an iterated integral. These properties are extensions of the formulas obtained by J. F. Steffensen (St., §§14, 25) for the mean integrals of a particular class of functions. The results of Steffensen were found in connection with some questions in statistics and insurance, and they show the advantage of the mean integral over the ordinary integral in applications to this type of problem.

The theorems are stated in terms of the S -integrals, and the corresponding properties for the S_m -integrals are given in the respective corollaries.

THEOREM 5. If the integrals $S \int_a^b \phi \, d\psi$, $S \int_a^b f \phi \, d\psi$, $S \int_a^b f \, d(\phi\psi)$ exist, and if $f(x)$ is bounded in (a, b) , then

$S \int_a^b f \psi \, d\phi$ exists, and

$$(8.1) \quad S \int_a^b f \psi \, d\phi = S \int_a^b f \, d(\phi\psi) - S \int_a^b f \phi \, d\psi.$$

Proof. By the substitution theorem,

$$(8.2) \quad S \int_a^b f \phi \, d\psi = S \int_a^b f \, d(S \int_a^x \phi \, d\psi)$$

and both integrals exist. By the integration by parts theorem, $S \int_a^x \phi \, d\psi$ exists ($a \leq x \leq b$), and

$$(8.3) \quad S \int_a^x \phi \, d\psi = [\phi\psi]_a^x - S \int_a^x \psi \, d\phi$$

$$d(S \int_a^x \phi \, d\psi) = d(\phi\psi) - d(S \int_a^x \psi \, d\phi).$$

Substituting from (8.3) into the right member of (8.2), we get

$$(8.4) \quad S \int_a^b f \phi \, d\psi = S \int_a^b f \, d(\phi\psi) - S \int_a^b f \, d(S \int_a^x \psi \, d\phi).$$

Since $S \int_a^x \psi \, d\phi$ exists, then from the substitution theorem, $S \int_a^b f \psi \, d\phi$ exists and equals $S \int_a^b f \, d(S \int_a^x \psi \, d\phi)$. Making use of this fact in (8.4), we get (8.1).

COROLLARY. Let $\phi(x)$ and $\psi(x)$ be functions of bounded variation in (a, b) , and let $f(x)$ be a bounded function which has no discontinuities in common with either $\phi(x)$ or $\psi(x)$. Then the theorem holds for the S_m -integral in both the N and the F senses.

The proof of the corollary is exactly the same as the proof of the theorem; for by virtue of Corollary 1, Theorem 4, the substitution theorem is valid for the Sm-integrals when these additional restrictions are placed on the functions.

THEOREM 6. Place $F(x) = S \int_a^x f \, d\phi$, $G(x) = S \int_x^b g \, d\psi$, with $F(a) = G(b) = 0$. If the integrals $S \int_a^b F g \, d\psi$, $S \int_a^b g \, d\psi$, and $S \int_a^b f \, d\phi$ exist, then $S \int_a^b G f \, d\phi$ exists, and¹

$$(8.5) \quad S \int_a^b (S \int_x^b g \, d\psi) f \, d\phi = S \int_a^b (S \int_a^x f \, d\phi) g \, d\psi.$$

Proof. From the substitution theorem, $S \int_a^b F \, d(S \int_a^x g \, d\psi)$ exists and equals $S \int_a^b F g \, d\psi$. Hence by the integration by parts theorem, $S \int_a^b (S \int_a^x g \, d\psi) \, dF$ exists, and

$$(8.6) \quad S \int_a^b (S \int_a^x g \, d\psi) \, dF = \left[F(x) \cdot S \int_a^x g \, d\psi \right]_a^b - S \int_a^b F g \, d\psi.$$

Substituting from

$$S \int_a^x g \, d\psi = S \int_a^b g \, d\psi - G(x)$$

into (8.6), we get

$$S \int_a^b g \, d\psi \, S \int_a^b dF - S \int_a^b G \, dF = F(b) \, S \int_a^b g \, d\psi - S \int_a^b F g \, d\psi,$$

from which

$$S \int_a^b G \, dF = S \int_a^b F g \, d\psi.$$

Since the integral on the right exists by hypothesis, then the integral on the left exists; and hence by the substitution theorem, $S \int_a^b G f \, d\phi$ exists and equals $S \int_a^b G \, dF$. Therefore

$$S \int_a^b G f \, d\phi = S \int_a^b F g \, d\psi,$$

which is equivalent to (8.5).

COROLLARY. Let $\phi(x)$ and $\psi(x)$ be functions of bounded variation, and let $f(x)$ and $g(x)$ be bounded functions which are both continuous where $\phi(x)$ and $\psi(x)$ have common

1. A result analogous to Theorem 6 has also been given by G. H. Hardy (Ha, p. 91), who derives the relation (8.5) by imposing on the functions restrictions which insure the existence of the integrals involved.

discontinuities. Then the theorem holds for the S_m -integrals in both the N and the F senses.

For by (VII), §6, $F(x)$ is bounded and is continuous where $\phi(x)$ is continuous, and $G(x)$ is bounded and is continuous where $\psi(x)$ is continuous; and hence the substitution theorem as applied in the above proof is valid for the S_m -integrals by Corollary 2 of Theorem 4.

The condition " $f(x)$ and $g(x)$ are both continuous where $\phi(x)$ and $\psi(x)$ have common discontinuities" may be replaced by " $f(x)$ has no discontinuities in common with $\psi(x)$, nor $g(x)$ with $\phi(x)$." The substitution theorem is then valid by virtue of Corollary 1 of Theorem 4.

The remark of L. M. Graves (Gr) on the relation between Dirichlet's formula and the integration by parts formula for Riemann integrals is also applicable here. For (8.5), which now plays the role of Dirichlet's formula, can be obtained from the integration by parts formula, and conversely. The analogy is not complete, however, since the derivation of (8.5) depends on the integration by parts theorem, and hence the two formulas are not equivalent.

CHAPTER IV

LINEAR FUNCTIONAL OPERATIONS ON FUNCTIONS
HAVING DISCONTINUITIES OF THE FIRST KIND¹

9. An extension of Riesz' Theorem. The most general form of a linear continuous functional operation on a function $f(x)$ continuous on (a,b) has been expressed by F. Riesz (Ri_2) as an NS-integral of $f(x)$ with respect to a function of bounded variation.² This result can be extended to the case where $f(x)$ has discontinuities of the first kind on (a,b) if the Dushnik or mean type of integral is used instead of the NS-integral. Moreover, the expression so obtained reduces to that given by Riesz when the function $f(x)$ is continuous.

The following definitions and properties enter into the statement and proof of the theorem. Let f denote an element of a linear vector space S of norm $\|f\|$, (see Hi_2 , §§2,3); then an operation T on S is linear if

$$(9.1) \quad T[k_1 f_1 + k_2 f_2] = k_1 T[f_1] + k_2 T[f_2],$$

where the k 's are any constants; and T is bounded if there exists an $M > 0$ depending on T such that

$$(9.2) \quad |T[f]| \leq M \|f\| \quad \text{for every } f \text{ of } S.$$

The condition of boundedness (9.2) is equivalent to the property of continuity

$$(9.3) \quad \lim_{n \rightarrow \infty} T[f_n] = T[f], \quad \text{when} \quad \lim_{n \rightarrow \infty} \|f_n - f\| = 0.$$

The class of all continuous functions (C) on (a,b) , and the class of functions having discontinuities of the first kind (D), constitute linear vector spaces whose norms are the maximum and least upper bound, respectively, of $|f(x)|$ on (a,b) .

An analogue of Riesz' Theorem for functions of class (D) is then contained in the

1. This chapter is printed, in a slightly modified form, in the Bulletin of the American Mathematical Society, v. 40 (1934), pp. 702-708.
2. Two new proofs of this theorem have been given recently by T. H. Hildebrandt and I. J. Schoenberg, (Hi_3 , pp. 317-319).

THEOREM 7. Every linear continuous operation T applied to a function $f(x)$ of (D) may be expressed in the form

$$(9.4) \quad T[f] = FS^* \int_a^b f \, d\psi + \sum_{i=1}^{\infty} [f(c_i) - f(c_i - 0)] \varphi(c_i),$$

where $c_0, c_1, c_2, \dots$, denote the points of discontinuity of $f(x)$, $c_0 = a$, $\psi(x)$ and $\varphi(x)$ depend only on T , and are such that $\psi(x)$ is of bounded variation, and $\varphi(x)$ vanishes except at a denumerable set of points and the sum of the absolute values of $\varphi(x)$ at these points is finite.

It will be noted that the expression for $T[f]$ depends on a subdivision of (a, b) into a set of open intervals and single points rather than on a subdivision into closed intervals as is the case when $f(x)$ is continuous and the ordinary integral is used. The first term on the right in (9.4) corresponds to the open intervals of the subdivision, and the second term arises from the points of discontinuity of $f(x)$. If $f(x)$ is continuous, the second term vanishes, and the expression for $T[f]$ reduces to the modified integral $FS^* \int_a^b f \, d\psi$. Now the NS-integral exists since $f(x)$ is continuous and $\psi(x)$ is of bounded variation, and since the FS^* -integral equals the NS-integral when the latter exists, it follows that the form for $T[f]$ in this case is the same as that obtained by Riesz.

To obtain the functions $\varphi(x)$ and $\psi(x)$ of (9.4), introduce the functions $g(t, x) = 1$ or 0 according as $t = x$ or $t \neq x$, and $h(t, x) = 1$ or 0 according as $a \leq t \leq x$ or $x < t \leq b$, and then place

$$(9.5) \quad \varphi(x) = T[g(t, x)], \quad \psi(x) = T[h(t, x)].$$

The properties of $\varphi(x)$ and $\psi(x)$ stated in the theorem follow from these definitions. Thus if $x_1, x_2, \dots, x_n$, is any finite set of points of (a, b) , then from (9.2) and the definition of $g(t, x)$,

$$\begin{aligned} \sum_{i=1}^n |\varphi(x_i)| &= \sum_{i=1}^n \operatorname{sgn} \varphi(x_i) T[g(t, x_i)] \\ &= M \left\| \sum_{i=1}^n \operatorname{sgn} \varphi(x_i) g(t, x_i) \right\| = M. \end{aligned}$$

Consequently $\varphi(x)$ vanishes except at a denumerable set of points, and the sum of the absolute values of $\varphi(x)$ at these points is $\leq M$. The function $\psi(x)$ is of bounded variation, since for any subdivision of (a, b) ,

$$\begin{aligned} \sum_{i=1}^n |\psi(x_i) - \psi(x_{i-1})| &= \sum_{i=1}^n |T[h(t, x_i) - h(t, x_{i-1})]| \\ &\leq M \left\| \sum_{i=1}^n [h(t, x_i) - h(t, x_{i-1})] \right\| = M. \end{aligned}$$

The given function $f(x)$ may be uniformly approximated on (a, b) by means of a sequence of functions of the type

$$(9.6) \quad \left\{ \begin{aligned} f_\sigma(x) &= \sum_{i=1}^n f(\xi_i) [h(t, x_i) - h(t, x_{i-1}) - g(t, x_i)] \\ &\quad + \sum_{i=0}^n f(x_i) g(t, x_i) \\ &= \sum_{i=1}^n f(\xi_i) [h(t, x_i) - h(t, x_{i-1})] + \\ &\quad \sum_{i=0}^n [f(x_i) - f(\xi_i)] g(t, x_i), \end{aligned} \right.$$

where $\sigma = (a=x_0 < x_1 < x_2 < \dots < x_n = b)$ is a finite subdivision of (a, b) , ξ_i is an arbitrary inner point of (x_{i-1}, x_i) , and $f(\xi_0) = 0$. For if $\epsilon > 0$ denotes an arbitrarily small quantity, then a subdivision σ_ϵ of (a, b) can be found (see H_1 , p. 217) so that the oscillation of $f(x)$ on the interior of any subinterval (x_{i-1}, x_i) , $(i=1, 2, \dots, n)$, is less than ϵ . Since $f_\sigma(x_i) = f(x_i)$, and $f_\sigma(x) = f(\xi_i)$ for $x_{i-1} < x < x_i$, $(i=1, 2, \dots, n)$, then if σ is a finer subdivision than σ_ϵ , $|f(x) - f_\sigma(x)| < \epsilon$ for any x on (a, b) , and hence $\lim_F f_\sigma(x) = f(x)$ uniformly. Therefore, from (9.3),

$$\lim_F T[f_\sigma(x)] = T[f(x)].$$

Applying T to (9.6), and making use of (9.5), we get

$$T[f_\sigma] = \sum_{i=1}^n f(\xi_i) [\psi(x_i) - \psi(x_{i-1})] + \sum_{i=0}^n [f(x_i) - f(\xi_i)] \varphi(x_i).$$

The limit of the first sum on the right exists (Du) and is equal to $FS^* \int_a^b f d\psi$. Therefore the limit of the second sum on the right exists, and we have

$$(9.7) \quad T[f] = FS^* \int_a^b f d\psi + \lim_F \sum_{i=0}^n [f(x_i) - f(\xi_i)] \varphi(x_i).$$

Let Q denote the sum on the right for a subdivision finer than σ_ϵ . Then $|f(x_i - 0) - f(\xi_i)| < \epsilon$, $(i=0, 1, 2, \dots, n)$, and hence

$$\begin{aligned} \left| Q - \sum_{i=0}^n [f(x_i) - f(x_{i-0})] \varphi(x_i) \right| &= \left| \sum_{i=0}^n [f(x_{i-0}) - f(\xi_i)] \varphi(x_i) \right| \\ &\leq \epsilon \sum_{i=0}^n |\varphi(x_i)| \leq \epsilon M. \\ \therefore \lim_F Q &= \lim_F \sum_{i=0}^n [f(x_i) - f(x_{i-0})] \varphi(x_i). \end{aligned}$$

Since $|f(c_i) - f(c_i - 0)| < \epsilon$ for every c_i which is not a point of σ_ϵ , then there is an integer $m \leq n$ such that, if Σ' denotes summation over the points of σ which are not points of discontinuity of $f(x)$,

$$\begin{aligned} & \left| \sum_{i=0}^n [f(c_i) - f(c_i-0)] \varphi(c_i) - \sum_{i=0}^n [f(x_i) - f(x_i-0)] \varphi(x_i) \right| = \\ & \left| \sum_{i=m}^n [f(c_i) - f(c_i-0)] \varphi(c_i) - \Sigma' [f(x_i) - f(x_i-0)] \varphi(x_i) \right| \leq \\ & \left| \sum_{i=m}^n |f(c_i) - f(c_i-0)| \varphi(c_i) \right| + \Sigma' |f(x_i) - f(x_i-0)| \varphi(x_i) \leq 2\epsilon M. \\ & \therefore \lim_F Q = \sum_{i=0}^{\infty} [f(c_i) - f(c_i-0)] \varphi(c_i). \end{aligned}$$

Substituting this value of $\lim Q$ into (9.7), we obtain (9.4), and thus the theorem is proved.

10. Alternate forms of the theorem. If the function $\chi(x) = T[k(t, x)]$, where $k(t, x) = 0$ or 1 according as $a \leq t < x$ or $x \leq t \leq b$, is used instead of $\psi(x)$, the expression

$$(10.1) \quad T[f] = FS^* \int_a^b f \, d\chi + \sum_{i=0}^{\infty} [f(c_i) - f(c_i+0)] \varphi(c_i)$$

is obtained instead of (9.4). To obtain a more general form of $T[f]$, which includes both (9.4) and (10.1), multiply (9.4) by λ and (10.1) by $1 - \lambda$, and add. Then

$$(10.2) \quad T[f] = FS^* \int_a^b f \, d\theta + \sum_{i=0}^{\infty} [f(c_i) - \{\lambda f(c_i-0) + (1-\lambda)f(c_i+0)\}] \varphi(c_i),$$

where $\theta(x) = \lambda \psi(x) + (1-\lambda) \chi(x)$. Certain conditions under which $T[f]$ can be expressed directly as a modified integral without any corrective terms being necessary may be read from (10.2). This is the case if T is such that $\varphi(x) \equiv 0$, or if¹ $f(x) = \lambda f(x-0) + (1-\lambda) f(x+0)$. In particular, $T[f]$ is expressible as a single integral if T operates on the class of functions $f(x)$ which are continuous on the right (or on the left) throughout (a, b) , or where $f(x) = 1/2 (f(x-0) + f(x+0))$ at every point.

Expressions for $T[f]$ corresponding to those given above can be obtained in terms of the FSM-integral; thus (9.4), (10.1), and (10.2) are equivalent, respectively, to

1. This case is considered by H. Hahn, (H_2 , p. 53).

$$(10.3) \quad T[f] = \text{FSm} \int_a^b f \, d\alpha + \sum_{i=0}^{\infty} [f(c_i) - f(c_{i-0})] \beta(c_i),$$

$$(10.4) \quad T[f] = \text{FSm} \int_a^b f \, d\alpha + \sum_{i=0}^{\infty} [f(c_i) - f(c_{i+0})] \beta(c_i),$$

$$(10.5) \quad T[f] = \text{FSm} \int_a^b f \, d\delta + \sum_{i=0}^{\infty} [f(c_i) - \{\lambda f(c_{i-0}) + (1-\lambda) f(c_{i+0})\}] \beta(c_i),$$

where

$$(10.6) \quad \begin{cases} \alpha(x) = 2\psi(x) - \psi(x+0), \quad \gamma(x) = 2\chi(x) - \chi(x-0), \\ \beta(x) = \varphi(x) - \frac{1}{2} [\psi(x+0) - \psi(x-0)] \\ \quad = \varphi(x) - \frac{1}{2} [\chi(x+0) - \chi(x-0)] \\ \delta(x) = \lambda\alpha(x) + (1-\lambda)\gamma(x). \end{cases}$$

To show the equivalence of (9.4) and (10.3), equate their right members when $f(x) = g(t, x)$ and when $f(x) = h(t, x)$. There results

$$\varphi(x) = \frac{1}{2} [\alpha(x+0) - \alpha(x-0)] + \beta(x),$$

$$\psi(x) - \psi(a) = \frac{1}{2} [\alpha(x+0) + \alpha(x)] - \alpha(a).$$

Since these relations hold for every x , then $\psi(a) = \alpha(a) = 0$ in the second equation, and hence

$$(10.7) \quad \begin{cases} \varphi(x) = \frac{1}{2} [\alpha(x+0) - \alpha(x-0)] + \beta(x) \\ \psi(x) = \frac{1}{2} [\alpha(x+0) + \alpha(x)]. \end{cases}$$

Since $\alpha(x)$ is continuous except at a denumerable set of points, it follows from the second of equations (10.7) that $\psi(x+0) = \alpha(x+0)$ and $\psi(x-0) = \alpha(x-0)$. By means of these relations, (10.7) can be reduced to the expressions given in (10.6) for $\alpha(x)$ and $\beta(x)$.

The equivalence of (10.1) and (10.4) is shown in the same way; and the equivalence of (10.2) and (10.5) follows from the way these equations were obtained. From (10.5) it is evident that a sufficient condition that $T[f]$ be expressible directly as a mean Stieltjes integral is that $\beta(x) \equiv 0$ or that $\varphi(x) = \frac{1}{2} (\psi(x+0) - \psi(x-0))$ at every point.

It is sometimes possible to express $T[f]$ in terms of the FYS-integral. By comparing the values of (9.4) and

$$(10.8) \quad T[f] = FYS \int_a^b f \, d\alpha$$

for the same specializations of the function $f(x)$ as above, it is found that the conditions for the equivalence of (9.4) and (10.8) are

$$\phi(x) = \psi(x+0) - \psi(x-0), \quad \psi(x) = \psi(x+0).$$

These conditions place a greater restriction on T than the corresponding conditions for expressing T as a single Stieltjes integral of the Dushnik or of the mean type. Consequently the Young-Stieltjes integral is not as well adapted to representing an operation $T[f]$ as these other modifications. The same remark applies to the Lebesgue-Stieltjes integral, which agrees with the FY-integral in this case.

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